

MAS212 Linear Algebra I Exercises 1 (with solutions)

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Assessed submission deadline 5pm on Friday 6th October 2006 in the blue box on the ground floor. Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together. Marks out of 100 are shown in square brackets after each question number.

1. [15] (a) Verify that the set of rational numbers, $\mathbb{Q}$, forms a group under addition by using the following definitions. A rational number is a quotient $\frac{p}{q}$ of integers p, q such that $q \neq 0$ and the sum of two rational numbers $\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$, where you may assume all the normal properties of integer arithmetic. *You may assume that $\mathbb{Q}$ is associative under addition.* (b) Does $\mathbb{Q}$ form a group under multiplication if the product of two rational numbers $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$? If not, give a simple way to construct a multiplicative group from $\mathbb{Q}$.

▼ Solution

Let $p, q, r, s \in \mathbb{Z}, q \neq 0, s \neq 0$, so that $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$.

(a) Then $\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$, where $ps + rq$ and qs are integers and $qs \neq 0$, so $\frac{p}{q} + \frac{r}{s} \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition.

$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} = \frac{rq + ps}{sq} = \frac{r}{s} + \frac{p}{q}$ by commutativity of integer arithmetic, so $\mathbb{Q}$ is commutative under addition.

$\frac{0}{1} \in \mathbb{Q}$ and $\frac{p}{q} + \frac{0}{1} = \frac{p}{q}$, so $\frac{0}{1} \in \mathbb{Q}$ is an additive identity element.

If $\frac{p}{q} \in \mathbb{Q}$ then $\frac{-p}{q} \in \mathbb{Q}$ and $\frac{p}{q} + \frac{-p}{q} = \frac{0}{1}$, so $\frac{-p}{q} \in \mathbb{Q}$ is the additive inverse of $\frac{p}{q} \in \mathbb{Q}$.

(b) $\frac{0}{1} \in \mathbb{Q}$ does not have a multiplicative inverse so $\mathbb{Q}$ does not form a multiplicative group.

However, $\mathbb{Q} \setminus \left\{ \frac{0}{1} \right\}$ does form a multiplicative group, i.e. just remove the additive identity element.

► Maple Solution

2. [20] Let a, b, c, x be elements of a field $\mathbb{K}$. Show *explicitly* how the field axioms allow the equation $a + bx = c$ to be solved for x and state any conditions that must be satisfied for a solution to exist.

▼ Solution

The field axioms assert that there exists an additive inverse of a , namely $-a$. Add this inverse to both sides of the equation. By additive closure the result is in $\mathbb{K}$; by additive commutativity and associativity it does not matter where in the sum we add it and we can group the terms as $(a + (-a)) + bx = c + (-a)$ to give $0 + bx = c + (-a)$. Since 0 is the additive identity we have $bx = c + (-a)$, which we conventionally write as $bx = c - a$.

The field axioms assert that provided $b \neq 0$ there exists a multiplicative inverse of b , namely b^{-1} . Multiply both sides of the equation by this inverse. By multiplicative closure the result is in $\mathbb{K}$; by multiplicative commutativity and associativity it does not matter where in the product we multiply it and we can group the terms as $(b b^{-1}) x = (c-a) b^{-1}$ to give $1 x = (c-a) b^{-1}$. Since 1 is the multiplicative identity we have $x = (c-a) b^{-1}$, **provided** $b \neq 0$, which we conventionally write as $x = \frac{(c-a)}{b}$.

3. [20] Consider the sets of 2×2 real matrices consisting of (a) all matrices, (b) all non-singular (i.e. invertible) matrices, (c) all singular (i.e. non-invertible) matrices, with standard matrix addition and multiplication. State whether each set is a field and if not state which field axioms fail. Where possible, illustrate each failure with a simple example.

Solution

In all cases the set is *not* a field because matrix multiplication does not generally commute. See Q4 for an example. But even if we ignore this (and so allow the set to be a *skew field* or *division algebra*) then ...

(a) No, because no singular matrix has a multiplicative inverse.

(b) No, because

(i) the additive identity (zero) matrix is not in the set (since it is singular);

(ii) hence the additive inverse is not defined;

(iii) addition of two non-singular matrices may produce a singular matrix, so the set is not closed under addition. Example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

(c) No, because

(i) the multiplicative identity (unit) matrix is not in the set (since it is non-singular);

(ii) hence the multiplicative inverse is not defined;

(iii) addition of two singular matrices may produce a non-singular matrix, so the set is not closed under addition. Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. [20] The matrices A and B over the Boolean field $\mathbb{F}_2$ are given by $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Compute $A + B$, AB and BA . Does multiplication of A and B commute? Compute A^2 and B^2 and deduce the values of A^n and B^n for all integer $n \geq 0$. [Remember that all your results must be over the Boolean field $\mathbb{F}_2$ represented by the set of elements $\{0, 1\}$, which you can achieve by computing over $\mathbb{Z}$ and then replacing all even integers by 0 and all odd integers by 1.]

Solution

$$A := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : B := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} :$$

$$A + B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \bmod 2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ hence } A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ over } \mathbb{F}_2.$$

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \bmod 2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ hence } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ over } \mathbb{F}_2.$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \bmod 2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \text{ hence } BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ over } \mathbb{F}_2.$$

Therefore, multiplication of A and B does not commute, since $AB \neq BA$.

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = A, \text{ hence } A^n = A \text{ for all integer } n \geq 0.$$

$$B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = B, \text{ hence } B^n = B \text{ for all integer } n \geq 0.$$

Such matrices are called *idempotent*.

► Maple Solution

5. [5] In any vector space V over a field $\mathbb{K}$ show, by using the vector space axioms explicitly, that $(a + b)(x + y) = ax + ay + bx + by$ for any $a, b \in \mathbb{K}$ and $x, y \in V$.

▼ Solution

Starting from the expression $(a + b)(x + y)$, apply distributivity over scalar addition to give $a(x + y) + b(x + y)$, then apply distributivity over vector addition to each term to give $ax + ay + bx + by$ (or vice versa).

6. [20] Let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$ be a set of vectors over the real field $\mathbb{R}$ with addition of elements of V defined coordinate-wise (exactly as in the vector space $\mathbb{R}^2$) and, for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, scalar multiplication defined by the (unusual) rule

$$c(a_1, a_2) = \begin{cases} (0, 0) & c = 0 \\ \left(\frac{a_1}{c}, \frac{a_2}{c}\right) & c \neq 0 \end{cases}.$$

Explain briefly why all the additive vector space axioms must be satisfied for V . However, V is not a vector space because one of the scalar multiplication axioms is not satisfied; by checking each of them determine which one.

▼ Solution

The vector space addition axioms are all satisfied because the definition of addition is the same as that for $\mathbb{R}^2$. Now consider scalar multiplication. Let $a = (a_1, a_2), b = (b_1, b_2) \in V$. Closure is satisfied since $ca = c(a_1, a_2) \in V$.

Distributivity over vector addition is satisfied since $c(a + b) = c(a_1 + b_1, a_2 + b_2) =$

$$\begin{cases} (0, 0) & c = 0 \\ \left(\frac{a_1 + b_1}{c}, \frac{a_2 + b_2}{c}\right) & c \neq 0 \end{cases} \text{ and}$$

$$ca + cb = c(a_1, a_2) + c(b_1, b_2)$$

$$= \begin{cases} (0, 0) + (0, 0) = (0, 0) & c = 0 \\ \left(\frac{a_1}{c}, \frac{a_2}{c} \right) + \left(\frac{b_1}{c}, \frac{b_2}{c} \right) = \left(\frac{a_1 + b_1}{c}, \frac{a_2 + b_2}{c} \right) & c \neq 0 \end{cases}$$

are equal.

Distributivity over scalar addition is **not** satisfied since, for $c, d \neq 0$,

$$(c + d) a = (c + d) (a_1, a_2) = \begin{cases} (0, 0) & c + d = 0 \\ \left(\frac{a_1}{c + d}, \frac{a_2}{c + d} \right) & c + d \neq 0 \end{cases} \text{ and}$$

$$c a + d a = c (a_1, a_2) + d (a_1, a_2) = \left(\frac{a_1}{c}, \frac{a_2}{c} \right) + \left(\frac{a_1}{d}, \frac{a_2}{d} \right) = \left(\left(\frac{1}{c} + \frac{1}{d} \right) a_1, \left(\frac{1}{c} + \frac{1}{d} \right) a_2 \right)$$

are **not** equal.

$$\text{Associativity is satisfied since } d(c(a)) = d(c(a_1, a_2)) = \begin{cases} (0, 0) & d c = 0 \\ \left(\frac{a_1}{d c}, \frac{a_2}{d c} \right) & d c \neq 0 \end{cases} \text{ and}$$

$$(dc)a = (dc) (a_1, a_2) = \begin{cases} (0, 0) & d c = 0 \\ \left(\frac{a_1}{d c}, \frac{a_2}{d c} \right) & d c \neq 0 \end{cases} \text{ are equal.}$$

Identity is satisfied since $1 (a_1, a_2) = (a_1, a_2)$.

Hence V is **not** a vector space.